

UPPER TAIL ESTIMATES FOR THE FIRST PASSAGE TIME IN THE FROG MODEL

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ABSTRACT. We study the so-called frog model with random initial configurations. The dynamics of this model are described as follows: Some particles are randomly assigned on any site of the multidimensional cubic lattice \mathbb{Z}^d ($d \geq 2$). Initially, only particles at the origin are active, and these independently perform simple random walks on \mathbb{Z}^d . Other particles are sleeping and do not move. When sleeping particles are attacked by an active particle, they become active and start moving in a similar fashion. In this context, the aim of this paper is to obtain some tail estimates for the first passage time at which an active particle reaches a target site. As a consequence, we derive order of intervals of initial positions of particles attaining the first passage time.

1. INTRODUCTION

1.1. The model. Let $d \geq 2$ and denote the d -dimensional cubic lattice by \mathbb{Z}^d . For a probability measure ν on the set $\mathbb{N}_0 := \{0, 1, 2, \dots\}$, we consider the probability space $\Omega := (\mathbb{N}_0)^{\mathbb{Z}^d}$ endowed with the usual product σ -field and the product measure $\mathbb{P} := \nu^{\otimes \mathbb{Z}^d}$. Denote an element of Ω by $\omega = (\omega(x))_{x \in \mathbb{Z}^d}$. Furthermore, write P^x for the law of the simple random walk on \mathbb{Z}^d starting at x . Let us introduce the following product measure on $\Sigma := (\mathbb{Z}^d)^{\mathbb{N}_0 \times \mathbb{Z}^d \times \mathbb{N}}$:

$$P^{\text{RWs}} := \left(\bigotimes_{x \in \mathbb{Z}^d} P^x \right)^{\otimes \mathbb{N}}.$$

Write $\mathbb{S} = (S_k(x, \ell))_{k \geq 0, x \in \mathbb{Z}^d, \ell \geq 1}$ for an element of Σ , i.e., under P^{RWs} , $S_\cdot(x, \ell)$ performs a simple random walk on \mathbb{Z}^d starting at x , independently of $(S_\cdot(y, \ell))_{y \neq x, \ell \geq 1}$.

For $\omega \in \Omega$, $\mathbb{S} \in \Sigma$ and $x, y \in \mathbb{Z}^d$, let us introduce the *first passage time* $T(x, y, \omega, \mathbb{S})$ from x to y as follows:

$$T(x, y, \omega, \mathbb{S}) := \inf \left\{ \sum_{m=0}^{n-1} t(x_m, x_{m+1}, \omega, \mathbb{S}); x = x_0, x_1, \dots, x_n = y \text{ for some } n \geq 1 \right\},$$

where for $a, b \in \mathbb{Z}^d$,

$$t(a, b, \omega, \mathbb{S}) := \inf \{k \geq 0; S_k(a, \ell) = b \text{ for some } 1 \leq \ell \leq \omega(a)\}$$

with the convention that $t(a, b, \omega, \mathbb{S}) := \infty$ if $\omega(a) = 0$. Throughout this paper, we drop ω and \mathbb{S} in the notation if there is no confusion. The main object is the first

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passage time $T(0, x)$ from the origin 0 to a target site x , conditioned on the event that $\omega(0) \geq 1$. Its intuitive meaning is as follows. We now regard simple random walks as “frogs”, and each site y of \mathbb{Z}^d has just $\omega(y)$ frogs (there may possibly be no frog at y). The condition that $\omega(0) \geq 1$ means that the origin has at least one frog. In this context, only frogs sitting on the origin are active, and these independently perform simple random walks. At first, other frogs are sleeping and do not move. When sleeping frogs are attacked by an active one, those become active and start moving in a similar fashion. Then, $T(0, x)$ describes the first passage time at which an active frog reaches a site x .

Let $\Omega_0 := \{\omega \in \Omega; \omega(0) \geq 1\}$, and we assume $\mathbb{P}(\Omega_0) > 0$ throughout this paper. For an initial configuration ω , denote by $\mathcal{P}(\omega)$ the set of all sites of \mathbb{Z}^d which are occupied by frogs, i.e., $\mathcal{P}(\omega) := \{x \in \mathbb{Z}^d; \omega(x) \geq 1\}$. Note that the first passage time has the subadditivity:

$$T(x, z) \leq T(x, y) + T(y, z), \quad x, y, z \in \mathbb{Z}^d.$$

As we are now working with the i.i.d. set-up of ω and \mathbb{S} , the subadditivity enables us to use the subadditive ergodic theorem to obtain the following proposition, which is called the *shape theorem*. For the proof, we refer the reader to [3, Theorem 1.1].

Proposition 1.1. *There exists a norm $\mu(\cdot)$ on \mathbb{R}^d (which is called the time constant) such that*

$$\lim_{\|x\| \rightarrow \infty} \frac{T(0, x) - \mu(x)}{\|x\|} = 0, \quad \mathbb{P} \otimes P^{\text{RWs}}(\cdot | \Omega_0)\text{-a.s.},$$

where $\|\cdot\|$ is the ℓ^1 -norm on \mathbb{R}^d . Furthermore, $\mu(\cdot)$ is invariant under permutations of the coordinates and under reflections in the coordinate hyperplanes, and always satisfies

$$(1.1) \quad \|x\| \leq \mu(x) \leq \mu(\xi_1)\|x\|,$$

where ξ_1 is the first coordinate vector of \mathbb{R}^d .

1.2. Main results. The following theorem gives an upper tail estimate for the first passage time.

Theorem 1.2. *There exist constants $0 < C_1, C_2, C_3 < \infty$ and $0 < \beta_1 < 1$ such that for all $x \in \mathbb{Z}^d \setminus \{0\}$ and $u \geq C_1\|x\|$,*

$$\mathbb{P} \otimes P^{\text{RWs}}(T(0, x) \geq u | \Omega_0) \leq C_2 e^{-C_3 u^{\beta_1}}.$$

This theorem derives a geometry of frogs. The following corollary says that each frog constructing $T(0, x)$ must find the next one within ℓ^1 -distance $\text{const} \times (\log \|x\|)^{1/\beta_1}$.

Corollary 1.3. *Suppose that $\mathbb{E}[\omega(0)] < \infty$. Then, there exist constants $0 < C_4, C_5 < \infty$ such that for all $N \in \mathbb{N}$ and $x \in \mathbb{Z}^d$ with $\|x\| \geq N$,*

$$(1.2) \quad \begin{aligned} & \mathbb{P} \otimes P^{\text{RWs}} \left(\exists v_1, v_2 \in \mathcal{P} \text{ with } \|v_1 - v_2\| \geq N \text{ such that } \left| \begin{array}{l} T(0, x) = T(0, v_1) + t(v_1, v_2) + T(v_2, x) \end{array} \right| \Omega_0 \right) \\ & \leq C_4 \|x\|^{2d} e^{-C_5 N^{\beta_1}}. \end{aligned}$$

In particular, there exists a constant $0 < C_6 < \infty$ such that with probability one, for all sufficiently large $x \in \mathbb{Z}^d$, if $n \geq 1$ and $0 = x_0, x_1, \dots, x_n = x$ satisfy $T(0, x) = \sum_{m=0}^{n-1} t(x_m, x_{m+1})$, then $\|x_m - x_{m+1}\| < C_6(\log \|x\|)^{1/\beta_1}$ for all $0 \leq m \leq n-1$.

Furthermore, combining Theorem 1.2 with some additional works, we obtain an upper tail estimate around the time constant.

Theorem 1.4. *There exists a constant $0 < \beta_2 < 1$ such that for all $\epsilon > 0$,*

$$\lim_{\|x\| \rightarrow \infty} \frac{1}{\|x\|^{\beta_2}} \log \mathbb{P} \otimes P^{\text{RWs}}(T(0, x) \geq (1 + \epsilon)\mu(x) | \Omega_0) < 0.$$

As stated in Proposition 2.1 below, Alves et al. [3] have already obtained an upper tail estimate very far away from the time constant. The goal of [3] is to prove Proposition 1.1 above. Its main difficulty is to check the integrability of the first passage time. For this end, it suffices to know the information of the very far upper tail for the first passage time. Theorems 1.2 and 1.4 extend their result to upper tail around the time constant, and, as in Corollary 1.3, our work may be useful for further investigation of the first passage time and its time constant.

1.3. Earlier literature. The original idea of the frog model provides a description of information spreading as follows. Consider that every active frog has some information. When it hits sleeping frogs, the former shares the information with the latter. Active frogs move freely and play a role in spreading the information.

The first published result on the frog model is due to Telcs–Wormald [19, Section 2.4], and this model was called the “egg model” there. They treated the frog model on \mathbb{Z}^d with one-frog-per-site initial configuration and proved that for all $d \geq 1$ it is recurrent, i.e., almost surely, the origin will be visited infinitely often by active frogs. (Otherwise, we call the frog model transient.) This result suggests an interesting relation between the strength of transience for a single random walk and the superior numbers of frogs.

To observe this more precisely, Popov [17] introduced the frog model with Bernoulli initial configuration and exhibited a threshold at which that frog model switches from transience to recurrence. After that, Alves et al. coped with such kind of problem for the frog model with random initial configuration and random lifetime. In this model, every frog may disappear, and it influences recurrence and transience, see [1, 18] for more details. In particular, [18] is a nice survey on the frog model and presents several open problems. It has also been a great help to recent progress on recurrence and transience for the frog model (refer the reader to [4, 5, 8, 10] for the frog model on lattices, [11, 12] for the frog model on trees, and [13] in a more general setting).

On the other hand, there are few results for the first passage time of the frog model except for [2, 3]. However, in the view of information spreading, it is important to investigate the first passage time more precisely. Accordingly, this article presents some upper tail estimates for the first passage time, and we study geometries of frogs constructing the first passage time.

1.4. Organization of the paper. Let us now describe how the present article is organized. In Section 2, for convenience, we summarize some notation and results

for the first passage time and supercritical site percolation on \mathbb{Z}^d . This is because, throughout this paper, we rely on the result obtained by Alves et al. [3] (see Proposition 2.1 below), and some exponential estimates and stochastic domination for the site percolation (see Propositions 2.2, 2.3 and 2.4).

The goal of Section 3 is to prove Theorem 1.2. Once it is done, Corollary 1.3 is a direct consequence. For the proof of Theorem 1.2, in Subsection 3.1, we introduce an event on which first passage times for crossing a large box are uniformly bounded by a constant (see the beginning of Subsection 3.1 for its precise definition). Proposition 3.1 says that this event induces a finitely dependent site percolation with parameter sufficiently close to one, and enables us to use a renormalization argument. This means that the first passage time from the origin to a target site is controlled by the chemical distance for the site percolation. Therefore, the theorem roughly speaking follows from an Antal–Pisztora type estimate.

In Section 4, we give the proof of Theorem 1.4. Let us briefly comment on it here. We basically follow the strategy taken in [6, Subsection 3.3]. Note that Proposition 1.1 suggests that if N is large enough, then at each site $y \in \mathbb{Z}^d$, it happens with high probability that $T(Ny, N(y + \xi)) \approx N\mu(\xi_1)$ for all $\xi \in \mathbb{Z}^d$ with $\|\xi\| = 1$. (However, $T(Ny, N(y + \xi)) = \infty$ holds if $\omega(Ny) = 0$. To avoid this, we modify the first passage time in Subsection 4.1.) Such a site y is called “good”, and good sites induce a finitely dependent site percolation with parameter sufficiently close to one (see Lemma 4.3 below). For simplicity, suppose that $x = n\xi_1$ and an arbitrary integer n is much larger than N . Results in Subsection 2.2 below guarantee that the failure probability of the following event decays exponentially in n : There exist good sites y_1, \dots, y_l such that

- $l \approx n/N$, and $\|y_i - y_{i+1}\| = 1$ for all $1 \leq i \leq l-1$,
- $\|Ny_1\|$ and $\|n\xi_1 - Ny_l\|$ are smaller than the order n .

It holds on this event that

$$\begin{aligned} T(0, n\xi_1) &\leq T(0, Ny_1) + \sum_{i=1}^{l-1} T(Ny_i, Ny_{i+1}) + T(Ny_l, x) \\ &\approx T(0, Ny_1) + \mu(n\xi_1) + T(Ny_l, x). \end{aligned}$$

Applying Theorem 1.2 to the first and third terms of the most right side, we can get the desired subexponential bound in the case $x = n\xi_1$. To carry out the above argument uniformly in all directions x , we need a few additional works combined with Lemma 4.2.

We close this section with some general notation. Denote by $\{\xi_1, \dots, \xi_d\}$ the canonical basis of \mathbb{R}^d . Write $\|\cdot\|$ for the ℓ^1 -norm on \mathbb{R}^d , and let $\mathcal{E}^d := \{\xi \in \mathbb{Z}^d; \|\xi\| = 1\}$. In addition, $B(x, r)$ is the ball of center $x \in \mathbb{R}^d$ and radius $r > 0$ for the ℓ^1 -norm, i.e.,

$$B(x, r) := \{y \in \mathbb{R}^d; \|y - x\| \leq r\}.$$

Throughout this paper, we use C_i and β_i , $i = 1, 2, \dots$, to denote constants with $0 < C_i < \infty$ and $0 < \beta_i < 1$, respectively.

2. PRELIMINARIES

In Section 2, we summarize some notation and results for the first passage time and supercritical site percolation on \mathbb{Z}^d .

2.1. Shifts and a known upper tail estimate for the frog model. For each $x \in \mathbb{Z}^d$, let $\theta_x : \Omega \times \Sigma \rightarrow \Omega \times \Sigma$ be the shift defined by

$$\theta_x(\omega, \mathbb{S}) := (\omega(\cdot + x), (S_k(\cdot + x, \ell) - x)_{k \geq 0, \ell \geq 1}).$$

We use the same notation for the restriction to Ω_0 , and introduce

$$\sigma^x(\omega) := \inf\{s \geq 1; \theta_x^s \omega \in \Omega_0\}.$$

Furthermore, set $\sigma_0^x(\omega) := 0$, and inductively,

$$\sigma_n^x(\omega) := \inf\{s > \sigma_{n-1}^x(\omega); \theta_x^s \omega \in \Omega_0\}, \quad n \geq 1.$$

The site $\sigma_n^x(\omega)x$ represents the position of the n -th intersection of $\mathcal{P}(\omega)$ with the half line $\mathbb{N}x$.

We state the result obtained by Alves et al. [3, Lemmata 2.2 and 2.3] in an appropriate form for our use.

Proposition 2.1. *There exist constants C_7 , C_8 and β_3 such that for all $x \in \mathbb{Z}^d$ and $u \geq \|x\|^4$,*

$$\mathbb{P} \otimes P^{\text{RWs}}(T(0, x) \geq u | \Omega_0) \leq C_7 e^{-C_8 u^{\beta_3}}.$$

2.2. Supercritical site percolation. We call $\gamma = (\gamma_0, \gamma_1, \dots, \gamma_l)$ a path if $\|\gamma_i - \gamma_{i+1}\| = 1$ for all $0 \leq i \leq l-1$, and write $\#\gamma$ for the length l of the path γ . Then, a subset A of \mathbb{Z}^d is said to be *connected* if any two sites in A are linked by a path using only sites in A .

Let $Y = (Y_v)_{v \in \mathbb{Z}^d}$ be a family of random variables taking values in $\{0, 1\}$. This induces the random subset $\{v \in \mathbb{Z}^d; Y_v = 1\}$ of \mathbb{Z}^d . The *chemical distance* $d_Y(x, y)$ for Y between x and y is defined as

$$d_Y(x, y) := \inf \left\{ \#\gamma; \gamma \text{ is a path from } x \text{ to } y \text{ using only sites in } \{v \in \mathbb{Z}^d; Y_v = 1\} \right\}.$$

If $\{v \in \mathbb{Z}^d; Y_v = 1\}$ contains an infinite connected component, then it is called an *infinite cluster* for Y . In particular, if there exists almost surely a unique infinite cluster for Y , then we denote it by $\mathcal{C}_\infty(Y)$.

Throughout this paper, for $0 < p < 1$, let $Z^p = (Z_v^p)_{v \in \mathbb{Z}^d}$ be a family of independent random variables satisfying

$$P(Z_v^p = 1) = 1 - P(Z_v^p = 0) = p, \quad v \in \mathbb{Z}^d,$$

which is the so-called *independent Bernoulli site percolation* of parameter p . It is well known that there exists $p_c = p_c(d) \in (0, 1)$ such that if $p > p_c$, we can find the infinite cluster $\mathcal{C}_\infty(Z^p)$ (see Theorems 1.10 and 8.1 of [9] for instance). The following proposition plays an important role to control the first passage time. See (2.2) and Corollary 2.2 of [7] for its proof.

Proposition 2.2. *For $p > p_c$, the following results (i) and (ii) hold:*

(1) *There exist constants C_9, C_{10}, C_{11} such that for all $x \in \mathbb{Z}^d$ and $u \geq C_9 \|x\|$,*

$$P(u \leq d_{Z^p}(0, x) < \infty) \leq C_{10} e^{-C_{11} u}.$$

(2) *There exist constants C_{12}, C_{13} such that for all $r > 0$,*

$$P(\mathcal{C}_\infty(Z^p) \cap B(0, r) = \emptyset) \leq C_{12} e^{-C_{13} r}.$$

To prove Theorem 1.4, we use the upper large deviation estimate for the chemical distance obtained by Garet–Marchand [6, Theorem 1.4]. Their argument works not only for bond percolation but also for site percolation.

Proposition 2.3. *For each $\alpha > 0$, there exists $p'(\alpha) \in (p_c, 1)$ such that for all $p > p'(\alpha)$,*

$$\limsup_{\|x\| \rightarrow \infty} \frac{1}{\|x\|} \log P((1 + \alpha)\|x\| \leq d_{Z^p}(0, x) < \infty) < 0.$$

We close this section with the concept of stochastic domination. Let $Y^{(1)} := (Y_v^{(1)})_{v \in \mathbb{Z}^d}$ and $Y^{(2)} := (Y_v^{(2)})_{v \in \mathbb{Z}^d}$ be families of random variables taking values in $\{0, 1\}$. We say that $Y^{(1)}$ *stochastically dominates* $Y^{(2)}$ if

$$E[f(Y^{(1)})] \geq E[f(Y^{(2)})]$$

for all bounded, increasing, measurable functions $f : \{0, 1\}^{\mathbb{Z}^d} \rightarrow \mathbb{R}$. Furthermore, a family $Y = (Y_v)_{v \in \mathbb{Z}^d}$ of random variables is said to be *finitely dependent* if there exists $L > 0$ such that any two sub-families $(Y_v)_{v \in \Lambda}$ and $(Y_v)_{v \in \Lambda'}$ are independent whenever $\Lambda, \Lambda' \subset \mathbb{Z}^d$ satisfy that $\|v - v'\| > L$ for all $v \in \Lambda, v' \in \Lambda'$.

The following stochastic comparison is frequently used in this paper. For the proof, we refer the reader to [9, Theorem 7.65] or [15, Theorem B26] for instance.

Proposition 2.4. *Suppose that $Y = (Y_v)_{v \in \mathbb{Z}^d}$ is a finitely dependent family of random variables taking values in $\{0, 1\}$. For a given $0 < p < 1$, if $\inf_{v \in \mathbb{Z}^d} P(Y_v = 1)$ is sufficiently close to one, then Y stochastically dominates Z^p .*

3. AN ANTAL–PISZTORA TYPE ESTIMATE

In this section, we will give the proofs of Theorem 1.2 and Corollary 1.3. Let us first construct a “good” renormalization enough to control passage times crossing large boxes.

3.1. Construction of crossings. Let N be a positive integer to be chosen large enough later. Once suitably chosen, N will be kept fixed throughout this section, and we omit it to lighten the notation. Write for $x \in \mathbb{Z}^d$ and $\xi \in \mathcal{E}^d$,

$$U_\xi(x) := \#\{1 \leq n \leq N; \theta_\xi^n \omega(x) \geq 1\}.$$

A site $v \in \mathbb{Z}^d$ is then said to be *white* if the following conditions (i)–(iv) hold:

- (1) For all $\xi \in \mathcal{E}^d$, $U_\xi(Nv) \geq 2$ holds.
- (2) For all $\xi \in \mathcal{E}^d$ and $0 \leq n < U_\xi(Nv)$,

$$\sigma_{n+1}^\xi \circ \theta_{Nv} - \sigma_n^\xi \circ \theta_{Nv} \leq \frac{1}{2} N^{1/4}.$$

(3) For all $\xi \in \mathcal{E}^d$ and $1 \leq n < U_\xi(Nv)$,

$$T(Nv + (\sigma_n^\xi \circ \theta_{Nv})\xi, Nv + (\sigma_{n+1}^\xi \circ \theta_{Nv})\xi) \leq N$$

and

$$T(Nv + (\sigma_{n+1}^\xi \circ \theta_{Nv})\xi, Nv + (\sigma_n^\xi \circ \theta_{Nv})\xi) \leq N.$$

(4) For all $\xi, \xi' \in \mathcal{E}^d$,

$$T(Nv + (\sigma^\xi \circ \theta_{Nv})\xi, Nv + (\sigma^{\xi'} \circ \theta_{Nv})\xi') \leq N.$$

Otherwise, v is called *black*.

The following propositions suggest that white sites induce a good renormalization to control passage times.

Proposition 3.1. *For $0 < p < 1$, there exists a positive integer N such that $(\mathbf{1}_{\{v \text{ is white}\}})_{v \in \mathbb{Z}^d}$ stochastically dominates Z^p . In particular, for p sufficiently close to one, we can find the infinite white cluster $\mathcal{C}_\infty^w := \mathcal{C}_\infty((\mathbf{1}_{\{v \text{ is white}\}})_{v \in \mathbb{Z}^d})$.*

Since we need some lemmata to prove Proposition 3.1, let us give its proof after that of Proposition 3.2.

Proposition 3.2. *Let $\gamma = (\gamma_0, \gamma_1, \dots, \gamma_l)$ be a white path. Suppose that $N\gamma_0 + s\xi, N\gamma_l + s'\xi' \in \mathcal{P}$ for some $1 \leq s, s' \leq N$ and $\xi, \xi' \in \mathcal{E}^d$. Then, one has*

$$(3.1) \quad T(N\gamma_0 + s\xi, N\gamma_l + s'\xi') \leq 5N^2l.$$

Proof. Let v_1 and v_2 be two adjacent white sites. Then, there exists $\eta \in \mathcal{E}^d$ such that $Nv_1 + (\sigma_n^\eta \circ \theta_{Nv_1})\eta = Nv_2 - (\sigma^{-\eta} \circ \theta_{Nv_2})\eta$, where $n := U_\eta(Nv_1)$. This together with the subadditivity implies that for all $1 \leq k, k' \leq N$ and $\zeta, \zeta' \in \mathcal{E}^d$ with $Nv_1 + k\zeta, Nv_2 + k'\zeta' \in \mathcal{P}$,

$$\begin{aligned} T(Nv_1 + k\zeta, Nv_2 + k'\zeta') &\leq T(Nv_1 + k\zeta, Nv_1 + (\sigma^\zeta \circ \theta_{Nv_1})\zeta) \\ &\quad + T(Nv_1 + (\sigma^\zeta \circ \theta_{Nv_1})\zeta, Nv_1 + (\sigma^\eta \circ \theta_{Nv_1})\eta) \\ &\quad + T(Nv_1 + (\sigma^\eta \circ \theta_{Nv_1})\eta, Nv_1 + (\sigma_n^\eta \circ \theta_{Nv_1})\eta) \\ &\quad + T(Nv_2 - (\sigma^{-\eta} \circ \theta_{Nv_2})\eta, Nv_2 + (\sigma^{\zeta'} \circ \theta_{Nv_2})\zeta') \\ &\quad + T(Nv_2 + (\sigma^{\zeta'} \circ \theta_{Nv_2})\zeta', Nv_2 + k'\zeta'). \end{aligned}$$

Since v_1 and v_2 are white, this is bounded by $5N^2$. Therefore, (3.1) immediately follows because $\#\gamma = l$. \square

We shall prepare some lemmata for the proof of Proposition 3.1.

Lemma 3.3. *For each $v \in \mathbb{Z}^d$, the event $\{v \text{ is white}\}$ depends only on states in $B(Nv, 2N)$.*

Proof. It suffices to check that for $1 \leq s, s' < N$ and $\xi, \xi' \in \mathcal{E}^d$, the event

$$(3.2) \quad \{T(Nv + s\xi, Nv + s'\xi') \leq N\}$$

depends only on states in $B(Nv, 2N)$. By the definition of the first passage time, (3.2) can be replaced with the event that there exist $n \geq 1$ and $x_0, x_1, \dots, x_n \in \mathbb{Z}^d$ with $x_0 = Nv + s\xi$ and $x_n = Nv + s'\xi'$ such that

$$\sum_{m=0}^{n-1} t(x_m, x_{m+1}) \leq N.$$

Note that the simple random walks can only move to an adjacent site at each step. This implies that if $\|x_m - x_0\| > N$ for some $1 \leq m \leq n$, then the above sum is strictly bigger than N . Therefore, x_1, \dots, x_n must be included in $B(Nv, 2N)$, and event (3.2) depends only on states in $B(Nv, 2N)$. \square

Lemma 3.4. *There exist constants C_{14} and C_{15} such that*

$$\begin{aligned} & \mathbb{P}\left(\exists \xi \in \mathcal{E}^d, \exists m \in [0, N) \text{ such that } \sigma_{m+1}^\xi - \sigma_m^\xi > \frac{1}{2}N^{1/4}\right) \\ & \leq C_{14}e^{-C_{15}N^{1/4}}. \end{aligned}$$

Proof. From Schwarz's inequality, the above probability is smaller than or equal to

$$\begin{aligned} (3.3) \quad & \sum_{\xi \in \mathcal{E}^d} \sum_{m=0}^{N-1} \mathbb{P}\left(\sigma_{m+1}^\xi - \sigma_m^\xi > \frac{1}{2}N^{1/4}\right) \\ & \leq \sum_{\xi \in \mathcal{E}^d} \sum_{m=0}^{N-1} \sum_{s=1}^{\infty} \mathbb{P}\left(\sigma^\xi \circ \theta_\xi^s > \frac{1}{2}N^{1/4}\right)^{1/2} \mathbb{P}(\sigma_m^\xi = s)^{1/2} \\ & \leq \mathbb{P}(\Omega_0^c)^{N^{1/4}/8} \sum_{\xi \in \mathcal{E}^d} \sum_{m=0}^{N-1} \sum_{s=1}^{\infty} \mathbb{P}(\sigma_m^\xi = s)^{1/2}. \end{aligned}$$

We have for all $\xi \in \mathcal{E}^d$ and $0 \leq m \leq N-1$,

$$\sum_{s=1}^{\infty} \mathbb{P}(\sigma_m^\xi = s)^{1/2} = \sum_{1 \leq s \leq 2\mathbb{P}(\Omega_0)^{-1}m} \mathbb{P}(\sigma_m^\xi = s)^{1/2} + \sum_{s > 2\mathbb{P}(\Omega_0)^{-1}m} \mathbb{P}(\sigma_m^\xi = s)^{1/2}.$$

It is clear that the first term of the right side is bounded by $2\mathbb{P}(\Omega_0)^{-1}m$. In addition, a standard large deviation estimate proves that for some constants C_{16} and C_{17} , the second term of the right side is smaller than or equal to

$$\begin{aligned} & \sum_{s \geq 2\mathbb{P}(\Omega_0)^{-1}m} \mathbb{P}\left(\frac{1}{s} \sum_{i=1}^s \mathbf{1}_{\{\omega(i\xi) \geq 1\}} = \frac{m}{s}\right)^{1/2} \\ & \leq \sum_{s \geq 2\mathbb{P}(\Omega_0)^{-1}m} \mathbb{P}\left(\frac{1}{s} \sum_{i=1}^s \mathbf{1}_{\{\omega(i\xi_1) \geq 1\}} \leq \frac{1}{2}\mathbb{P}(\Omega_0)\right)^{1/2} \leq C_{16}e^{-C_{17}m}. \end{aligned}$$

With these observations, the summations in the most right side of (3.3) is of at most order N^2 , and the lemma immediately follows. \square

Lemma 3.5. *There exist constants C_{18} and C_{19} such that for all $\xi \in \mathcal{E}^d$ and $1 \leq n < N$,*

$$\mathbb{P} \otimes P^{\text{RWs}}(T(\sigma_n^\xi, \sigma_{n+1}^\xi) \vee T(\sigma_{n+1}^\xi, \sigma_n^\xi) \geq N) \leq C_{18} e^{-C_{19} N^{\beta_3 \wedge (1/4)}}.$$

Proof. From Lemma 3.4, the above probability is smaller than or equal to

$$C_{14} e^{-C_{15} N^{1/4}} + \mathbb{P} \otimes P^{\text{RWs}}\left(T(\sigma_n^\xi, \sigma_{n+1}^\xi) \vee T(\sigma_{n+1}^\xi, \sigma_n^\xi) \geq N, \sigma_{n+1}^\xi - \sigma_n^\xi \leq \frac{1}{2} N^{1/4}\right).$$

From Schwarz's inequality and the translation invariance for θ_ξ , the last probability is bounded by

$$\begin{aligned} & \mathbb{P} \otimes P^{\text{RWs}}\left(T(\sigma_n^\xi, \sigma_{n+1}^\xi) \vee T(\sigma_{n+1}^\xi, \sigma_n^\xi) \geq N, \sigma_{n+1}^\xi - \sigma_n^\xi \leq \frac{1}{2} N^{1/4}\right) \\ & \leq \mathbb{P} \otimes P^{\text{RWs}}\left(T(0, \sigma^\xi) \vee T(\sigma^\xi, 0) \geq N, \sigma^\xi \leq \frac{1}{2} N^{1/4}, 0 \in \mathcal{P}\right)^{1/2} \\ & \quad \times \sum_{s=1}^{\infty} \mathbb{P}(\sigma_n^\xi = s)^{1/2}. \end{aligned}$$

As in the proof of Lemma 3.4, the above sum is of at most order N . On the other hand, Proposition 2.1 proves that

$$\begin{aligned} & \mathbb{P} \otimes P^{\text{RWs}}\left(T(0, \sigma^\xi) \vee T(\sigma^\xi, 0) \geq N, \sigma^\xi \leq \frac{1}{2} N^{1/4}, 0 \in \mathcal{P}\right) \\ & \leq C_7 N^{1/4} e^{-C_8 N^{\beta_3}}. \end{aligned}$$

With these observations, the lemma follows. \square

Lemma 3.6. *There exist constants C_{20} , C_{21} such that for all $\xi, \xi' \in \mathcal{E}^d$,*

$$\mathbb{P} \otimes P^{\text{RWs}}(T(\sigma^\xi, \sigma^{\xi'}) \geq N) \leq C_{20} e^{-C_{21} N^{\beta_3 \wedge (1/4)}}.$$

Proof. From Lemma 3.4,

$$\begin{aligned} & \mathbb{P} \otimes P^{\text{RWs}}(T(\sigma^\xi, \sigma^{\xi'}) \geq N) \\ & \leq C_{14} e^{-C_{15} N^{1/4}} + \sum_{1 \leq s, s' \leq N^{1/4}/2} \mathbb{P} \otimes P^{\text{RWs}}(T(0, s'\xi' - s\xi) \geq N | \Omega_0). \end{aligned}$$

From Proposition 2.1, the last sum is bounded by $(N^{1/2}/4) C_7 e^{-C_8 N^{\beta_3}}$, and the lemma follows. \square

We are now in a position to prove Proposition 3.1.

Proof of Proposition 3.1. The proof is done by using Proposition 2.4, Lemma 3.3 and the same strategy taken in the proof of Proposition 5.2 of [16]. Hence, our task is now to check that $\inf_{v \in \mathbb{Z}^d} \mathbb{P} \otimes P^{\text{RWs}}(v \text{ is black})$ converges to zero as $N \rightarrow \infty$. To do this, from Lemma 3.3 and the translation invariance for θ_{Nv} , it suffices to check that $\mathbb{P} \otimes P^{\text{RWs}}(0 \text{ is black})$ tends to zero as $N \rightarrow \infty$. Note that

$$\mathbb{P}(\exists \xi \in \mathcal{E}^d \text{ such that } U_\xi(0) < 2) \leq 4dN\mathbb{P}(\Omega_0^c)^{N-1},$$

and we use Lemmata 3.4, 3.5 and 3.6 imply that

$$\begin{aligned} \mathbb{P} \otimes P^{\text{RWs}}(0 \text{ is black}) &\leq 4dN\mathbb{P}(\Omega_0^c)^{N-1} + C_{14}e^{-C_{15}N^{1/4}} \\ &\quad + 2dC_{18}e^{-C_{19}N^{\beta_3 \wedge (1/4)}} + (2d)^2C_{20}e^{-C_{21}N^{\beta_3 \wedge (1/4)}}. \end{aligned}$$

Therefore, $\mathbb{P} \otimes P^{\text{RWs}}(0 \text{ is black})$ converges to zero as $N \rightarrow \infty$. \square

3.2. Proofs of Theorem 1.2 and Corollary 1.3.

Proof of Theorem 1.2. Take $p > p_c$ sufficiently close to one as in Proposition 3.1. Let $x \in \mathbb{Z}^d \setminus \{0\}$ and $v(x)$ a site in \mathbb{Z}^d satisfying that $Nv(x)$ is the closest to x for the ℓ^1 -norm, with a deterministic rule to break ties. Thanks to (2) of Proposition 2.2 and Proposition 2.4, one has

$$\begin{aligned} (3.4) \quad &\mathbb{P} \otimes P^{\text{RWs}}(\mathcal{C}_\infty^w \cap B(0, u^{1/4}) = \emptyset \text{ or } \mathcal{C}_\infty^w \cap B(v(x), u^{1/4}) = \emptyset) \\ &\leq P(\mathcal{C}_\infty(Z^p) \cap B(0, u^{1/4}) = \emptyset \text{ or } \mathcal{C}_\infty(Z^p) \cap B(v(x), u^{1/4}) = \emptyset) \\ &\leq 2C_{12}e^{-C_{13}u^{1/4}}. \end{aligned}$$

Note that for $u \geq d\|x\|$, $v_1 \in B(0, u^{1/4})$ and $v_2 \in B(v(x), u^{1/4})$,

$$\|v_1 - v_2\| \leq 2u^{1/4} + \frac{1}{N}\|Nv(x) - x\| + \frac{\|x\|}{N} \leq 4u.$$

Let $d^w(\cdot, \cdot)$ be the chemical distance for $(\mathbf{1}_{\{v \text{ is white}\}})_{v \in \mathbb{Z}^d}$. Then, (i) of Proposition 2.2 and Proposition 2.4 prove that there exists a constant C_{22} such that for all $u \geq d\|x\|$,

$$\begin{aligned} (3.5) \quad &\mathbb{P} \otimes P^{\text{RWs}}\left(\begin{array}{l} \exists v_1 \in \mathcal{C}_\infty^w \cap B(0, u^{1/4}), \exists v_2 \in \mathcal{C}_\infty^w \cap B(v(x), u^{1/4}) \\ \text{such that } d^w(v_1, v_2) \geq 4C_9u \end{array}\right) \\ &\leq P\left(\begin{array}{l} \exists v_1 \in \mathcal{C}_\infty(Z^p) \cap B(0, u^{1/4}), \exists v_2 \in \mathcal{C}_\infty(Z^p) \cap B(v(x), u^{1/4}) \\ \text{such that } d_{Z^p}(v_1, v_2) \geq 4C_9u \end{array}\right) \\ &\leq C_{22}u^{d/2}e^{-4C_9C_{11}u}. \end{aligned}$$

Moreover, Proposition 2.1 shows that for some constant C_{23} ,

$$\begin{aligned} (3.6) \quad &\mathbb{P} \otimes P^{\text{RWs}}\left(\begin{array}{l} \exists y \in B(0, 2Nu^{1/4}) \cap \mathcal{P}, \exists z \in B(Nv(x), 2Nu^{1/4}) \cap \mathcal{P} \\ \text{such that } T(0, y) \geq (3N)^4u \text{ or } T(z, x) \geq (3N)^4u \end{array} \middle| \Omega_0\right) \\ &\leq C_{23}(2N)^{2d}u^{d/2}e^{-C_8(3N)^4\beta_3u^{\beta_3}}. \end{aligned}$$

We now consider the event F that

- there exist $v_1 \in \mathcal{C}_\infty^w \cap B(0, u^{1/4})$ and $v_2 \in \mathcal{C}_\infty^w \cap B(v(x), u^{1/4})$ such that $d^w(v_1, v_2) < 4C_9u$,
- $T(0, y) < (3N)^4u$ and $T(z, x) < (3N)^4u$ hold for all $y \in B(0, 2Nu^{1/4}) \cap \mathcal{P}$ and $z \in B(Nv(x), 2Nu^{1/4}) \cap \mathcal{P}$.

From Proposition 3.2, on the event $F \cap \Omega_0$,

$$T(0, x) < 2(3N)^4u + 20N^2C_9u < (3N)^5C_9u.$$

Therefore, the theorem is a direct consequence of (3.4), (3.5) and (3.6). \square

Proof of Corollary 1.3. Thanks to Theorem 1.2, the left side of (1.2) is smaller than or equal to

$$\begin{aligned}
 & C_2 e^{-C_3(C_1\|x\|)^{\beta_1}} \\
 & + \sum_{\substack{v_1, v_2 \in B(0, C_1\|x\|) \\ \|v_1 - v_2\| \geq N}} \mathbb{P} \otimes P^{\text{RWs}}(t(0, v_2 - v_1) = T(0, v_2 - v_1) | \Omega_0) \\
 (3.7) \quad & \leq C_2 e^{-C_3(C_1\|x\|)^{\beta_1}} + \sum_{\substack{v_1, v_2 \in B(0, C_1\|x\|) \\ \|v_1 - v_2\| \geq N}} \{I_1(v_2 - v_1) + I_2(v_2 - v_1)\},
 \end{aligned}$$

where for $y \in \mathbb{Z}^d$,

$$\begin{aligned}
 I_1(y) &:= \mathbb{P} \otimes P^{\text{RWs}} \left(\max_{\substack{0 \leq k \leq C_1\|y\| \\ 1 \leq \ell \leq \omega(0)}} \|S_k(0, \ell)\| \geq \|y\| \middle| \Omega_0 \right), \\
 I_2(y) &:= \mathbb{P} \otimes P^{\text{RWs}} \left(\max_{\substack{0 \leq k \leq C_1\|y\| \\ 1 \leq \ell \leq \omega(0)}} \|S_k(0, \ell)\| < \|y\|, t(0, y) = T(0, y) \middle| \Omega_0 \right).
 \end{aligned}$$

To estimate $I_1(v_2 - v_1)$ and $I_2(v_2 - v_1)$, we rely on the following simple large deviation estimate for the simple random walk, see [14, Lemma 1.5.1] for instance: For any $a > 0$, there exists a constant C_{24} , which may depend on a , such that for all $n, u \geq 0$,

$$P^{\text{RWs}} \left(\max_{0 \leq k \leq n} \|S_k(0, 1)\| \geq au\sqrt{n} \right) \leq C_{24} e^{-u}.$$

Taking $a = C_1^{-1/2}$, $n = C_1\|v_2 - v_1\|$ and $u = \|v_2 - v_1\|^{1/2}$ for our use, one has

$$\begin{aligned}
 & I_1(v_2 - v_1) \\
 & \leq \mathbb{P}(\Omega_0)^{-1} \mathbb{E} \left[\mathbf{1}_{\Omega_0} \sum_{\ell=1}^{\omega(0)} P^{\text{RWs}} \left(\max_{0 \leq k \leq C_1\|v_2 - v_1\|} \|S_k(0, \ell)\| \geq \|v_2 - v_1\| \right) \right] \\
 & \leq \mathbb{P}(\Omega_0)^{-1} \mathbb{E}[\omega(0)] C_{24} e^{-\|v_2 - v_1\|^{1/2}}.
 \end{aligned}$$

On the other hand, we also apply Theorem 1.2 to $I_2(v_2 - v_1)$:

$$\begin{aligned}
 & I_2(v_2 - v_1) \\
 & \leq \mathbb{P} \otimes P^{\text{RWs}}(t(0, v_2 - v_1) > C_1\|v_2 - v_1\|, t(0, v_2 - v_1) = T(0, v_2 - v_1) | \Omega_0) \\
 & \leq C_2 e^{-C_3(C_1\|v_2 - v_1\|)^{\beta_1}}.
 \end{aligned}$$

It follows that the right side of (3.7) is smaller than or equal to

$$\begin{aligned}
 & C_2 e^{-C_3(C_1\|x\|)^{\beta_1}} \\
 & + \sum_{\substack{v_1, v_2 \in B(0, C_1\|x\|) \\ \|v_1 - v_2\| \geq N}} \{ \mathbb{P}(\Omega_0)^{-1} \mathbb{E}[\omega(0)] C_{24} e^{-\|v_2 - v_1\|^{1/2}} + C_2 e^{-C_3(C_1\|v_2 - v_1\|)^{\beta_1}} \},
 \end{aligned}$$

and (1.2) immediately follows. The second assertion of Corollary 1.3 is a direct consequence of (1.2) and the Borel–Cantelli lemma. \square

4. THE UPPER TAIL ESTIMATE AROUND THE TIME CONSTANT

This section gives the proof of Theorem 1.4. We basically follow the strategy taken in [6, Subsection 3.3]. To apply a renormalization procedure, let us modify the first passage time in Subsection 4.1. The proof of Theorem 1.4 is done in Subsection 3.2.

4.1. The modified first passage time. For any $x \in \mathbb{Z}^d$, let x^* be the closest point to x in \mathcal{P} for the ℓ^1 -norm, with a deterministic rule to break ties. We define $T^*(x, y) := T(x^*, y^*)$, and it satisfies the subadditivity:

$$T^*(x, z) \leq T^*(x, y) + T^*(y, z), \quad x, y, z \in \mathbb{Z}^d.$$

The following proposition says that T^* is a modification of T which keeps the time constant.

Proposition 4.1. *For each $x \in \mathbb{Z}^d$, $\mathbb{P} \otimes P^{\text{RWs}}$ -a.s. and in $L^1(\mathbb{P} \otimes P^{\text{RWs}})$,*

$$\begin{aligned} \mu(x) &= \lim_{n \rightarrow \infty} \frac{1}{n} T^*(0, nx) \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \mathbb{E} \otimes E^{\text{RWs}}[T^*(0, nx)] = \inf_{n \geq 1} \frac{1}{n} \mathbb{E} \otimes E^{\text{RWs}}[T^*(0, nx)]. \end{aligned}$$

Proof. Once we check the existence of the limit

$$\begin{aligned} \mu^*(x) &:= \lim_{n \rightarrow \infty} \frac{1}{n} T^*(0, nx) \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \mathbb{E} \otimes E^{\text{RWs}}[T^*(0, nx)] = \inf_{n \geq 1} \frac{1}{n} \mathbb{E} \otimes E^{\text{RWs}}[T^*(0, nx)], \end{aligned}$$

the proposition immediately follows. Indeed, Proposition 1.1 shows that we have on the event Ω_0 with positive $\mathbb{P} \otimes P^{\text{RWs}}$ -probability,

$$\mu(x) = \lim_{n \rightarrow \infty} \frac{1}{\sigma_n^x} T(0, \sigma_n^x x) = \lim_{n \rightarrow \infty} \frac{1}{\sigma_n^x} T^*(0, \sigma_n^x x) = \mu^*(x).$$

We apply the subadditive ergodic theorem to the process $T^*(mx, nx)$, $0 \leq m < n$, $m, n \in \mathbb{N}_0$. As this process is stationary with respect to the shift θ_x and satisfies the subadditivity, it suffices to check the integrability of $T^*(0, x)$ and the ergodicity of the process $T^*(nkx, (n+1)kx)$, $n \geq 1$, for each $k \geq 1$.

Let us first show the integrability. To do this,

$$\begin{aligned} &\mathbb{E} \otimes E^{\text{RWs}}[T^*(0, x)] \\ &\leq \int_0^\infty \mathbb{P}\left(\|0^*\| > \frac{1}{3}u^{1/4}\right) du + \int_0^\infty \mathbb{P}\left(\|x - x^*\| > \frac{1}{3}u^{1/4}\right) du \\ &\quad + \int_0^\infty \mathbb{P} \otimes P^{\text{RWs}}\left(T^*(0, x) \geq u, \|0^*\| \leq \frac{1}{3}u^{1/4}, \|x - x^*\| \leq \frac{1}{3}u^{1/4}\right) du. \end{aligned}$$

It is clear that the first and second terms in the right side are bounded. Moreover, Proposition 2.1 shows that for some constant C_{25} , the third term is smaller than or equal to

$$(3\|x\|)^4 + \int_{(3\|x\|)^4}^\infty C_{25} \left(\frac{1}{3}u^{1/4}\right)^{2d} C_7 e^{-C_8 u^{\beta_3}} du < \infty,$$

and the integrability follows.

We next prove that for each $k \geq 1$, the process $T^*(nkx, (n+1)kx)$, $n \geq 1$, is ergodic. To do this, let φ be the canonical shift on $\mathbb{R}^{\mathbb{N}_0}$. Our goal is to show that the shift φ is ergodic with respect to $\mathbb{P} \otimes P^{\text{RWs}}(T^*(\cdot kx, (\cdot+1)kx) \in \cdot)$. Assume that A satisfies

$$\mathbb{P} \otimes P^{\text{RWs}}(T^*(\cdot kx, (\cdot+1)kx) \in A\Delta\varphi^{-1}(A)) = 0.$$

Setting $B := \{T^*(\cdot kx, (\cdot+1)kx) \in A\}$, one has

$$\mathbb{P} \otimes P^{\text{RWs}}(B\Delta\theta_{kx}^{-1}(B)) = \mathbb{P} \otimes P^{\text{RWs}}(T^*(\cdot kx, (\cdot+1)kx) \in A\Delta\varphi^{-1}(A)) = 0.$$

This means that B is a θ_{kx} -invariant set under $\mathbb{P} \otimes P^{\text{RWs}}$. Since θ_{kx} is ergodic, we have $\mathbb{P} \otimes P^{\text{RWs}}(B) \in \{0, 1\}$. \square

4.2. Proofs of Theorem 1.4. Let us first prepare some notation and lemmata. We denote by \mathcal{S}_d the symmetric group on $\{1, \dots, d\}$. For each $x = (x_1, \dots, x_d) \in \mathbb{R}^d$, $\sigma \in \mathcal{S}_d$ and $\epsilon \in \{+1, -1\}^d$, we define

$$\Psi_{\sigma, \epsilon}(x) := \sum_{i=1}^d \epsilon(i)x_{\sigma(i)}\xi_i.$$

Then, $\mathcal{O}(\mathbb{Z}^d) := \{\Psi_{\sigma, \epsilon}; \sigma \in \mathcal{S}_d, \epsilon \in \{+1, -1\}^d\}$ is the group of orthogonal transformations that preserve the grid \mathbb{Z}^d . Consequently, its elements also preserve the time constant $\mu(\cdot)$.

To study the first passage time in a given direction x , we want to find a basis of \mathbb{R}^d adapted to the studied direction, i.e., made of images of x by elements of $\mathcal{O}(\mathbb{Z}^d)$. The following technical lemma, which is obtained by Garet–Marchand [6, Lemma 2.2], gives the existence of such a basis.

Lemma 4.2. *There exists a constant C_{26} such that for each $x \in \mathbb{R}^d$, we can find a family $(g_{1,x}, g_{2,x}, \dots, g_{d,x}) \in (\mathcal{O}(\mathbb{Z}^d))^d$ with $g_{1,x} = \text{Id}_{\mathbb{R}^d}$ and the linear map $L_x : \mathbb{R}^d \rightarrow \mathbb{R}^d$ defined by*

$$L_x(\xi_i) = g_{i,x}(x), \quad 1 \leq i \leq d,$$

and satisfying

$$C_{26}\|x\|\|y\| \leq \|L_x(y)\| \leq \|x\|\|y\|, \quad y \in \mathbb{R}^d.$$

We break the proof of Theorem 1.4 into three steps. Fix an arbitrary $\epsilon > 0$.

Step 1. In this step, we choose appropriate constants for our proof. Note that by (1.1), $\mu(\cdot) \geq 1$ on the compact set $\partial B(0, 1)$, where $\partial B(0, 1)$ denotes the topological boundary of $B(0, 1)$. Then, there exists $\delta > 0$ such that for all $y \in \partial B(0, 1)$,

$$(4.1) \quad \left(1 + \frac{3\delta}{2C_1}\right)(1 + \delta)^2\mu(y) + 2\delta < \mu(\xi)(1 + \epsilon)$$

and

$$\delta < \frac{C_1}{2},$$

where C_1 is the constant appearing in Theorem 1.2. To shorten notation, write

$$\alpha := \frac{\delta}{2C_1} < \frac{1}{4}.$$

Take an integer $M \geq 4$ large enough to have

$$(4.2) \quad M \geq \frac{d}{\delta} \max \left\{ \frac{\mu(\xi_1)}{2}, \frac{8C_1}{C_{26}} \right\}.$$

Step 2. We shall construct a “good” renormalization which derives a good behavior of the first passage time. Let N be a positive integer to be chosen large enough later. A site $v \in \mathbb{Z}^d$ is said to be *good* if the following conditions (i) and (ii) hold for all $y \in \frac{1}{M}\mathbb{Z}^d \cap \partial B(0, 1)$:

- (1) $T^*(NL_{My}(v), NL_{My}(v + \xi)) \leq NM\mu(y)(1 + \delta)$ for all $\xi \in \mathcal{E}^d$.
- (2) $(NL_{My}(v))^*$ is included in $B(NL_{My}(v), \sqrt{N})$ and $(NL_{My}(v + \xi))^*$ belongs to $B(NL_{My}(v + \xi), \sqrt{N})$ for all $\xi \in \mathcal{E}^d$.

Otherwise, v is called *bad*.

Lemma 4.3. *For each $0 < p < 1$, there exists a positive integer N such that $(\mathbf{1}_{\{v \text{ is good}\}})_{v \in \mathbb{Z}^d}$ stochastically dominates Z^p . In particular, for p sufficiently close to one, we can find the infinite good cluster $\mathcal{C}_\infty^g := \mathcal{C}_\infty((\mathbf{1}_{\{v \text{ is good}\}})_{v \in \mathbb{Z}^d})$.*

Proof. As in the proof of Proposition 3.1, it suffices to check that $\mathbb{P} \otimes P^{\text{RWs}}(0 \text{ is bad})$ tends to zero as $N \rightarrow \infty$, and the family $(\mathbf{1}_{\{v \text{ is good}\}})_{v \in \mathbb{Z}^d}$ is finitely dependent. Since the set $\frac{1}{M}\mathbb{Z}^d \cap \partial B(0, 1)$ is finite, Proposition 4.1 implies that $\mathbb{P} \otimes P^{\text{RWs}}(0 \text{ is bad})$ converges to zero as $N \rightarrow \infty$. Moreover, Lemma 4.2 ensures that if $\|v - v'\| > (4/C_{26})\mu(\xi_1)(1 + \delta)$, then for all $y \in \frac{1}{M}\mathbb{Z}^d \cap \partial B(0, 1)$,

$$\|NL_{My}(v) - NL_{My}(v')\| > 4MN\mu(\xi_1)(1 + \delta).$$

This means that $(\mathbf{1}_{\{v \text{ is good}\}})_{v \in \mathbb{Z}^d}$ is finitely dependent. \square

From now on, we pick p such that the second assertion of Lemma 4.3 is established and

$$(4.3) \quad p > p' \left(\frac{\alpha}{1 + 2\alpha} \right) > p_c,$$

where $p'(\cdot)$ is the parameter appearing in Proposition 2.3.

Lemma 4.4. *Let $y \in \frac{1}{M}\mathbb{Z}^d \cap \partial B(0, 1)$ and $\gamma = (\gamma_0, \gamma_1, \dots, \gamma_l)$ a good path. Then, one has*

$$T^*(NL_{My}(\gamma_0), NL_{My}(\gamma_l)) \leq NM\mu(y)(1 + \delta)l.$$

Proof. By the subadditivity,

$$T^*(NL_{My}(\gamma_0), NL_{My}(\gamma_l)) \leq \sum_{i=0}^{l-1} T^*(NL_{My}(\gamma_i), NL_{My}(\gamma_{i+1})).$$

Since the path γ only uses good sites, this is smaller than or equal to $NM\mu(y)(1 + \delta)l$. \square

For an arbitrary $x \in \mathbb{Z}^d \setminus \{0\}$, let $x' := x/\|x\|$. Then, we can take $\hat{x} \in \frac{1}{M}\mathbb{Z}^d \cap \partial B(0, 1)$ with $\|x' - \hat{x}\| \leq d/(2M)$. In particular, by (4.2),

$$\|x' - \hat{x}\| \leq \frac{C_{26}\alpha}{8}$$

and

$$(4.4) \quad |\mu(x') - \mu(\hat{x})| \leq \mu(\xi_1) \|x' - \hat{x}\| \leq \mu(\xi_1) \frac{d}{2M} \leq \delta.$$

Lemma 4.2 tells us that for all $1 \leq i \leq d$,

$$\mu(L_{M\hat{x}}(\xi_i)) = M\mu(\hat{x}), \quad \|L_{M\hat{x}}(\xi_i)\| = M,$$

and for all $y \in \mathbb{R}^d$,

$$C_{26}M\|y\| \leq \|L_{M\hat{x}}(y)\| \leq M\|y\|.$$

Denote by $d^g(\cdot, \cdot)$ the chemical distance for $(\mathbf{1}_{\{v \text{ is good}\}})_{v \in \mathbb{Z}^d}$, and consider the event

$$G := \{\exists v \in \mathcal{A}(0, \alpha\|\bar{x}\|), \exists w \in \mathcal{A}(\bar{x}, \alpha\|\bar{x}\|) \text{ such that } d^g(v, w) \leq (1 + 3\alpha)\|\bar{x}\|\},$$

where

$$\bar{x} := \left\lfloor \frac{\|x\|}{MN} \right\rfloor \xi_1,$$

and

$$\mathcal{A}(z, r) := \{y \in \mathbb{Z}^d; r/2 \leq \|y - z\| \leq r\}, \quad z \in \mathbb{Z}^d, r > 0.$$

We use Lemma 4.3 to obtain

$$\begin{aligned} \mathbb{P} \otimes P^{\text{RWs}}(G^c) &\leq \mathbb{P} \otimes P^{\text{RWs}}\left(\forall v \in \mathcal{A}(0, \alpha\|\bar{x}\|), \forall w \in \mathcal{A}(\bar{x}, \alpha\|\bar{x}\|) \right. \\ &\quad \left. \text{suth that } d_{Z^p}(v, w) > (1 + 3\alpha)\|\bar{x}\| \right) \\ &\leq 2P(B(0, \alpha\|\bar{x}\|/2) \cap \mathcal{C}_\infty(Z^p) = \emptyset) \\ &\quad + \sum_{\substack{v \in B(0, \alpha\|\bar{x}\|) \\ w \in B(\bar{x}, \alpha\|\bar{x}\|)}} P\left(\frac{1 + 3\alpha}{1 + 2\alpha} \|v - w\| < d_{Z^p}(v, w) < \infty\right). \end{aligned}$$

From (2) of Proposition 2.2,

$$P(B(0, \alpha\|\bar{x}\|/2) \cap \mathcal{C}_\infty(Z^p) = \emptyset) \leq C_{12}e^{-C_{13}\alpha\|\bar{x}\|/2}.$$

In addition, Proposition 2.3 and (4.3) guarantee that there exist constants C_{27} and C_{28} such that for all $v \in B(0, \alpha\|\bar{x}\|)$ and $w \in B(\bar{x}, \alpha\|\bar{x}\|)$,

$$P\left(\frac{1 + 3\alpha}{1 + 2\alpha} \|v - w\| < d_{Z^p}(v, w) < \infty\right) \leq C_{27}e^{-C_{28}(1-2\alpha)\|\bar{x}\|}.$$

These bounds, combined with the definition of \bar{x} , imply that for some constants C_{29} and C_{30} ,

$$(4.5) \quad \begin{aligned} \mathbb{P} \otimes P^{\text{RWs}}(G^c) &\leq C_{29}e^{-C_{13}\alpha\|x\|/(2MN)} \\ &\quad + C_{29}\left(\frac{\alpha\|x\|}{MN}\right)^{2d} e^{-C_{30}(1-2\alpha)\|x\|/(MN)}. \end{aligned}$$

Step 3. Let us complete the proof. Without loss of generality, we assume that

$$(4.6) \quad \|x\| \geq \frac{4MN}{\alpha C_{26}}.$$

By the definition of x' and (4.1),

$$(4.7) \quad \begin{aligned} & \mathbb{P} \otimes P^{\text{RWs}}(T(0, x) \geq (1 + \epsilon)\mu(x)|\Omega_0) \\ &= \mathbb{P} \otimes P^{\text{RWs}}(T(0, x) \geq \mu(x')(1 + \epsilon)\|x\||\Omega_0) \\ &\leq \mathbb{P} \otimes P^{\text{RWs}}\left(T(0, x) > \left(1 + \frac{3\delta}{2C_1}\right)(1 + \delta)^2\mu(x')\|x\| + 2\delta\|x\| \middle| \Omega_0\right). \end{aligned}$$

On the event Ω_0 , we now suppose that

- the event G occurs,
- $T(0, y) < \delta\|x\|$ holds for all $y \in NL_{M\hat{x}}(\mathcal{A}(0, \alpha\|\bar{x}\|)) + B(0, \sqrt{N})$,
- $T(z, x) < \delta\|x\|$ holds for all $z \in [NL_{M\hat{x}}(\mathcal{A}(\bar{x}, \alpha\|\bar{x}\|)) + B(0, \sqrt{N})] \cap \mathcal{P}$.

Thanks to Lemma 4.4 and (4.4), one has for some $v \in \mathcal{A}(0, \alpha\|\bar{x}\|)$ and $w \in \mathcal{A}(\bar{x}, \alpha\|\bar{x}\|)$,

$$\begin{aligned} T(0, x) &\leq T(0, (NL_{M\hat{x}}(v))^*) \\ &\quad + T^*(NL_{M\hat{x}}(v), NL_{M\hat{x}}(w)) + T((NL_{M\hat{x}}(w))^*, x) \\ &\leq \left(1 + \frac{3\delta}{2C_1}\right)(1 + \delta)^2\mu(x')\|x\| + 2\delta\|x\|. \end{aligned}$$

Hence, the most right side of (4.7) is bounded from above by

$$\begin{aligned} & \mathbb{P}(\Omega_0)^{-1} \mathbb{P} \otimes P^{\text{RWs}}(G^c) \\ &+ \sum_{y \in NL_{M\hat{x}}(\mathcal{A}(0, \alpha\|\bar{x}\|)) + B(0, \sqrt{N})} \mathbb{P} \otimes P^{\text{RWs}}(T(0, y) \geq \delta\|x\||\Omega_0) \\ &+ \sum_{z \in NL_{M\hat{x}}(\mathcal{A}(\bar{x}, \alpha\|\bar{x}\|)) + B(0, \sqrt{N})} \mathbb{P} \otimes P^{\text{RWs}}(T(z, x) \geq \delta\|x\|, z \in \mathcal{P}|\Omega_0). \end{aligned}$$

By (4.5), the first term is harmless. Our task is now to estimate the second and third terms. We use Lemma 4.2 and (4.6) to obtain that for $y \in NL_{M\hat{x}}(\mathcal{A}(0, \alpha\|\bar{x}\|)) + B(0, \sqrt{N})$,

$$\|y\| \leq MN\alpha\|\bar{x}\| + \sqrt{N} \leq \alpha\|x\| + \sqrt{N} \leq \frac{17}{16}\alpha\|x\|,$$

and

$$\begin{aligned} \|y\| &\geq MNC_{26} \frac{\alpha\|\bar{x}\|}{2} - \sqrt{N} \\ &\geq C_{26} \frac{\alpha\|x\|}{2} - MNC_{26} \frac{\alpha}{2} - \sqrt{N} \geq \frac{13}{32}C_{26}\alpha\|x\|. \end{aligned}$$

Similarly, for $z \in NL_{M\hat{x}}(\mathcal{A}(\bar{x}, \alpha\|\bar{x}\|)) + B(0, \sqrt{N})$,

$$\frac{13}{32}C_{26}\alpha\|x\| \leq \|z - NL_{M\hat{x}}(\bar{x})\| \leq \frac{17}{16}\alpha\|x\|.$$

This proves that

$$\begin{aligned}
\|x - z\| &\leq \|x - NL_{M\hat{x}}(\bar{x})\| + \|NL_{M\hat{x}}(\bar{x}) - z\| \\
&\leq \|x\|\|x' - \hat{x}\| + MN + \frac{17}{16}\alpha\|x\| \\
&\leq 2\alpha\|x\| = \frac{\delta}{C_1}\|x\|,
\end{aligned}$$

and

$$\begin{aligned}
\|x - z\| &\geq \|z - NL_{M\hat{x}}(\bar{x})\| - \|NL_{M\hat{x}}(\bar{x}) - x\| \\
&\geq \frac{13}{32}C_{26}\alpha\|x\| - \frac{3}{8}C_{26}\alpha\|x\| = \frac{C_{26}}{32}\alpha\|x\|.
\end{aligned}$$

With these observations, Theorem 1.2 implies that there exist constants C_{31} and C_{32} such that

$$\begin{aligned}
&\sum_{y \in NL_{M\hat{x}}(\mathcal{A}(0, \alpha\|\bar{x}\|)) + B(0, \sqrt{N})} \mathbb{P} \otimes P^{\text{RWs}}(T(0, y) > \delta\|x\| | \Omega_0) \\
&\leq \sum_{(13/32)C_{26}\alpha\|x\| \leq \|y\| \leq (17/16)\alpha\|x\|} \mathbb{P} \otimes P^{\text{RWs}}\left(T(0, y) > \frac{4\delta}{5\alpha}\|y\| \mid \Omega_0\right) \\
&\leq \sum_{(13/32)C_{26}\alpha\|x\| \leq \|y\| \leq (17/16)\alpha\|x\|} \mathbb{P} \otimes P^{\text{RWs}}(T(0, y) > C_1\|y\| | \Omega_0) \\
&\leq C_{31}(\alpha\|x\|)^d e^{-C_{32}(\alpha\|x\|)^{\beta_1}}
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{z \in NL_{M\hat{x}}(\mathcal{A}(\bar{x}, \alpha\|\bar{x}\|)) + B(0, \sqrt{N})} \mathbb{P} \otimes P^{\text{RWs}}(T(z, x) > \delta\|x\|, z \in \mathcal{P} | \Omega_0) \\
&\leq \sum_{(C_{26}/32)\alpha\|x\| \leq \|z - x\| \leq 2\alpha\|x\|} \mathbb{P} \otimes P^{\text{RWs}}(T(0, x - z) > C_1\|x - z\| | \Omega_0) \\
&\leq C_{31}(\alpha\|x\|)^d e^{-C_{32}(\alpha\|x\|)^{\beta_1}}.
\end{aligned}$$

Therefore, the proof is complete. \square

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